

# Flocks and Fluids: Studying Alignment Dynamics with a PDE Toolbox

Trevor M. Leslie

IIT Applied Math Research Collaboratorium  
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# Collective Dynamics

Small-scale interactions

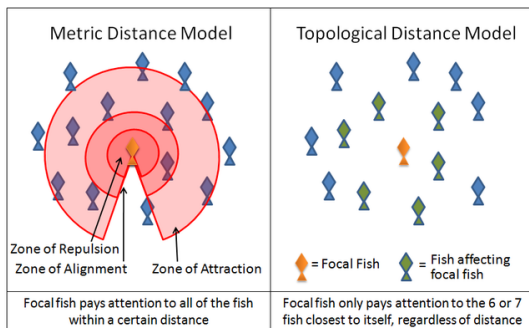
*How?*  
 $\Rightarrow$

Large-scale structures



**Figure:** A flock of starlings. (Public domain photo, from Wikipedia)

# Some Modeling Considerations



Left: Reynolds 3-Zone model (1986); implemented in computer animation (“Boids”).

Right: Topological Distance model. Relevance experimentally verified by the StarFlag project.

Other possible considerations: Physical external forces, Leaders within a group, Predators, etc., etc., etc.

(public domain figure from Wikipedia)

# The Cucker–Smale System

The **Cucker–Smale** system of ODE's is a simple model that incorporates *only* the alignment forcing between agents (RHS of 2nd equation).

$$\begin{cases} \dot{x}_i = v_i \\ m_i \dot{v}_i = \sum_{j=1}^N m_i m_j \phi(x_i, x_j)(v_j - v_i) \end{cases}$$

- $(m_i, x_i, v_i) = (\text{mass, position, velocity})$  of *i*th *agent*
- $\phi = \text{communication protocol}$ . Often  $\phi(x_i, x_j) = \phi(|x_i - x_j|)$  in the literature, for simplicity.
- Many variations on C–S model exist. One example:  $\phi(x_i, x_j)$  replaced with a more general ‘topological’ protocol.

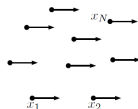
# Typical Features of Long-Time Dynamics

Expected long-time dynamics for C–S under ‘nice’ initial conditions have features like the following. (Question: When exactly does this happen?)

## Velocity Alignment:

$$\lim_{t \rightarrow +\infty} \mathcal{V}(t) = 0,$$

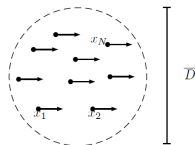
$$(\mathcal{V}(t) = \max_{i,j} |v_i(t) - v_j(t)|).$$



## Flocking:

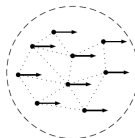
$$\sup_{t \geq 0} \mathcal{D}(t) \leq \overline{\mathcal{D}} < +\infty$$

$$(\mathcal{D}(t) = \max_{i,j} |x_i(t) - x_j(t)|).$$



## Strong Flocking:

$$x_i(t) - x_j(t) \rightarrow \overline{x_{ij}}, \text{ for all } i, j.$$



# The Cucker–Smale and Euler Alignment Systems

**Cucker–Smale** (CS) system:

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**Kinetic Cucker–Smale** (KCS) via mean-field limit ( $N \rightarrow \infty$ ):

$$\partial_t f + \operatorname{div}_x(vf) + \operatorname{div}_v Q(f, f) = 0.$$

$$Q(f, f)(x, v, t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(|x - y|)(w - v)f(x, v, t)f(y, w, t) \, dy \, dw.$$

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**Euler Alignment System (EA) for  $\rho = \int f dv$  and  $\rho u = \int fv dv$ :**

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + \mathcal{R}) = \int_{\mathbb{R}^d} \rho(x) \rho(y) \phi(x - y)(\mathbf{u}(y) - \mathbf{u}(x)) dy. \end{cases}$$

# Forward and Inverse Problems

Most questions about (CS) and related models that one can try to study fall into one of two broad categories:

- ★ (Forward Problem) Given  $\phi$  (or a class of  $\phi$ ), what do the predicted dynamics look like, esp. after a long time?

Plenty of interesting problems even for (CS), but most natural setting is (KCS) or (EA).

- (Inverse Problem) Given dynamics, what is the best choice for  $\phi$ ?

Ming Zhong, Fei Lu work on this from machine learning perspective.  
One challenge: Lack of available real-world data.

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- Common Paper Topics
  - Feature Y of the density profile/agent configuration can be predicted from Feature X of the initial conditions.

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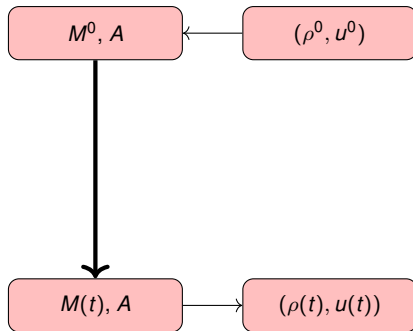
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- Common Tools (again, depends on  $\phi$ )

- Theory of Parabolic PDE’s or Hyperbolic Conservation Laws
- Fluid Dynamics or Continuum Mechanics—a lot carries over!
- Mathematical Analysis Tools (E.g., from Math 500 and Math 501!)

# Schematic for a Recent Project:

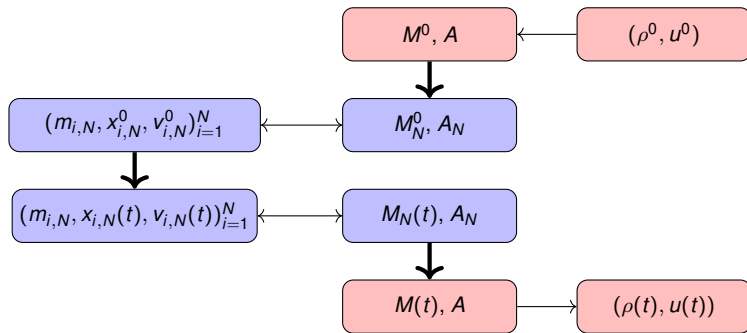
Measure-Valued Solutions for 1D Euler Alignment System



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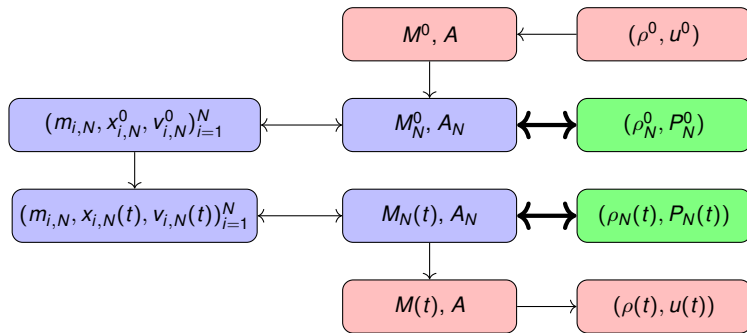
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# Schematic for a Recent Project:

Measure-Valued Solutions for 1D Euler Alignment System



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2. **Discretization:** **Sticky particle Cucker–Smale dynamics** provide a front-tracking approximation scheme for solutions of the scalar balance law.
3. **Approximation by Atomic (EA):** Atomic density and momentum measures generated by atomic initial data approximate the ‘entropically selected’ solution.

# This looks interesting—How do I get involved?

What you need:

- Really solid Analysis skills. (The more you know, the better—but mastery of the basics is more important than familiarity with any given topic.)
- Enthusiasm for learning a lot about ODE's, PDE's, and mathematical fluid/continuum mechanics
- Drive and Patience

Want to get involved?

- Consider taking a reading course with me to see if we would work well together.

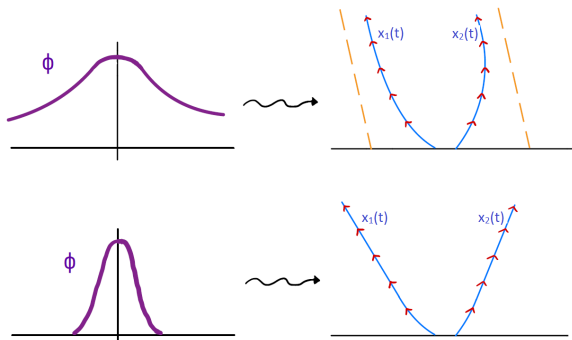
Thanks for your attention!

[https://www.youtube.com/watch?v=V4f\\_1\\_r80RY](https://www.youtube.com/watch?v=V4f_1_r80RY)

## Bonus Slides

# Heuristics: Long and Short-Range Interactions

Slowly-decaying protocols  $\phi$  ensure that agents never escape the range of influence of one another, promoting flocking.



If  $\phi$  is *heavy-tailed*,  $\int_1^\infty \phi(r) dr = +\infty$ , then (strong) flocking occurs exponentially fast for any initial data.

(Cucker–Smale 2007, Ha–Tadmor 2008, Ha–Liu 2008, Dietert–Shvydkoy 2021)

# Heuristics: Propagation Through Chain Connectivity

Systems subject to compactly supported communication protocols must rely on propagation of information through chains of agents.

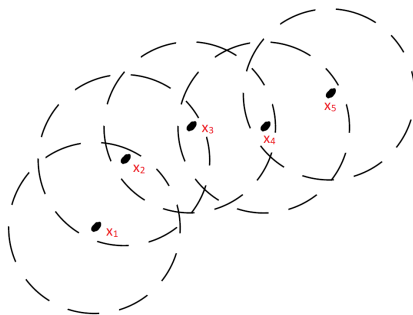
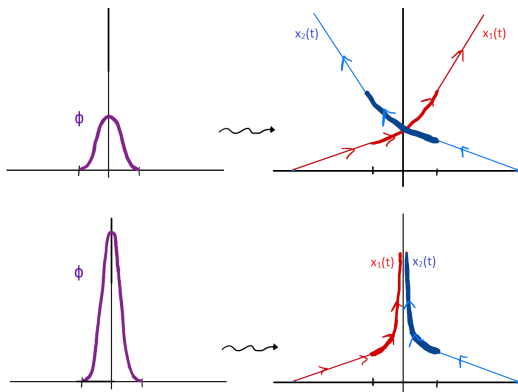


**Figure:** Agents  $x_1$  and  $x_5$  are not in direct communication, but they feel each others' influence through the 'chain' of communication. They lose contact once this chain is broken.

Despite the fragility of chain connectivity, strong enough local interactions, plus (somewhat restrictive) connectivity assumptions on initial data, are sufficient to guarantee flocking. (Shvydkoy–Tadmor 2020, Morales–Peszek–Tadmor 2020)

# Heuristics: Short-Range Communication and Collision-Avoidance

In addition to promoting flocking, strong local communication also promotes collision avoidance.



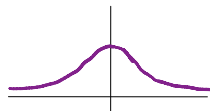
If  $\phi(r) \sim r^{-s}$ ,  $s \geq 1$ , *no* collisions can occur for Cucker–Smale.

(Carrillo–Choi–Mucha–Peszek 2017, Dietert–Shvydkoy 2021)

# Euler Alignment with Bounded and Singular Protocols

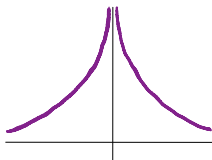
Euler Alignment System:

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$\phi$  bounded

- System has a *hyperbolic* character.
- Problem makes sense on  $\mathbb{R}^d$  or  $\mathbb{T}^d$ .



$\phi(r) \sim \frac{1}{r^s}$ , near  $r = 0$ , with  $s \in (d, d + 2)$ :

- System has a *parabolic* character; driven by density.
- Vacuum can lead to blowup (Arnaiz–Castro 2018)
- Existing results mostly posed on  $\mathbb{T}^d$

Wellposedness results are mostly 1D or ‘close’ to 1D.

Many papers of the form: “Assume a regular solution exists for (EA). Under assumptions X, Y, Z, flocking occurs.” Techniques are similar to C–S flocking.