

Modeling Interacting Particle Systems using Distribution Dependent Stochastic Differential Equations (SDEs)

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Mean Field Motivation

- Consider the system of 2 SDEs in $\mathbb{R}$:

$$dX^1(t) = -(aX^1(t) - b\mu(t))dt + dW^1(t), \quad X^1(0) = x_1$$

$$dX^2(t) = -(aX^2(t) - b\mu(t))dt + dW^2(t), \quad X^2(0) = x_2$$

$$\mu(t) = \frac{X^1(t) + X^2(t)}{2}$$

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- Now imagine we have n SDEs in $\mathbb{R}$:

$$dX^{n,i}(t) = -(aX^{n,i}(t) - b\mu^n(t))dt + dW^i(t), \quad X^{n,i}(0) = x_i$$

$$\mu^n(t) = \frac{1}{n} \sum_{k=1}^n X^{n,k}(t)$$

n -Particle System

- More generally we have n SDEs in $\mathbb{R}^d$

$$\begin{aligned}dX^{n,i}(t) &= b(X^{n,i}(t), \mu^n(t))dt + \sigma(X^{n,i}(t), \mu^n(t))dW^i(t), \\ \mu^n(t) &= \frac{1}{n} \sum_{k=1}^n \delta_{X^{n,k}(t)} \quad X_0^{n,i} = \xi^i\end{aligned}\tag{0.1}$$

n -Particle System

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- ▶ $X^{n,i}(t)$ is the position of the i th particle at time t in $\mathbb{R}^d$
- ▶ W_t^i are independent standard Brownian motions
- ▶ ξ^i i.i.d (independent and identically distributed).
- ▶ $\delta_{X^{n,i}(t)}$ is the dirac measure on $X^{n,i}(t)$, so $\mu^n(t)$ is the empirical measure

McKean-Vlasov SDE

Let $n \rightarrow \infty$ in

$$dX^{n,i}(t) = b(X^{n,i}(t), \mu^n(t))dt + \sigma(X^{n,i}(t), \mu^n(t))dW_t^i,$$

$$\mu^n(t) = \frac{1}{n} \sum_{k=1}^n \delta_{X^{n,k}(t)} \quad X_0^{n,i} = \xi_0^i$$

- ▶ $\mu^n(t) = \frac{1}{n} \sum_{k=1}^n \delta_{X^{n,k}(t)}$ contribution vanishes for an individual particle
- ▶ Each particle becomes asymptotically i.i.d
- ▶ By the strong law of large numbers $\mu^n(t) \rightarrow \mu(t)$ for a deterministic measure flow $\mu(t)$
- ▶ We can expect that $\mu(t)$ is the law of $X^i(t)$, since the empirical measure of i.i.d. samples converges to the true distribution

McKean-Vlasov SDE

- ▶ As $n \rightarrow \infty$, we expect (0.1) to converge to

$$\begin{aligned}dX(t) &= b(X(t), \mu(t))dt + \sigma(X(t), \mu(t))dW_t, \\ \mu(t) &= \mathcal{L}(X(t)) \quad X_0 = \xi\end{aligned}\tag{0.2}$$

- ▶ (0.2) is called the McKean-Vlasov stochastic differential equation (MVSDE)

Example

- ▶ Consider the SDE which represents a "modified" OU-Process or Langvin equation:

$$dX(t) = -(aX(t) - b\mathbb{E}[X(t)])dt + \sigma dW(t)$$

- ▶ Application to finance in modeling similar stock prices
- ▶ Application to physics in modeling a harmonic oscillator

Interesting Questions

We can investigate the following questions (or many others):

- ▶ Does the particle system converge to the SDE? If so, in what sense?
- ▶ Does the SDE have a solution?
- ▶ Does the process converge? If so, in what sense?
- ▶ What is the rate of convergence?
- ▶ ...